A novel 15-vertex solution of the Sutherland equation

1989 J. Phys. A: Math. Gen. 22 L25

(http://iopscience.iop.org/0305-4470/22/1/005)

J. Phys. A: Math. Gen. 22 (1989) L25-31. Printed in the UK

LETTER TO THE EDITOR

A novel 15-vertex solution of the Sutherland equation

Bernd Gölzer

Fachrichtung 11.1—Theoretische Physik, Universität des Saarlandes, D-6600 Saarbrücken, Federal Republic of Germany

Received 10 August 1988

Abstract. The Sutherland equation, implying exact integrability of vertex models, is studied on a two-dimensional square lattice. A novel 15-vertex solution commuting with a spin-1 Hamiltonian is constructed.

In the theory of exactly soluble models in statistical physics specific significance has been attributed to solutions of the equation

$$[A_nA_{n+1}, H_{n,n+1}] = B_nA_{n+1} - A_nB_{n+1}. \tag{1}$$

Equation (1) will be called the Sutherland equation throughout this letter. This equation (Sutherland 1970) is the local condition for a transfer matrix $T = \mathrm{Tr}\,\Pi_n A_n$ of a vertex model on a planar lattice to commute with the Hamilton operator $\mathscr{H} = \Sigma_n H_{n,n+1}$ of a one-dimensional spin chain with nearest-neighbour interaction, i.e. that

$$[T, \mathscr{H}] = 0 \tag{2}$$

holds.

Equation (2) ensures that T and $\mathscr{H}$ have a common system of eigenfunctions. Apart from some trivial cases these are generalised Bethe ansatz eigenfunctions for all solutions studied below. A set of operators $\{A_n, B_n, H_{n,n+1}\}$ satisfying equation (1) will be called a solution of the Sutherland equation. Four different solutions have been reported for matrix dimension two (Kasteleyn 1975). These four solutions represent some of the most thoroughly studied models in two-dimensional statistical mechanics; briefly they are:

(i) the free-fermion vertex model commuting with the XY Hamiltonian;

(ii) the general six-vertex model commuting with the XXZ Hamiltonian with an additional non-Hermitian term;

(iii) the symmetric eight-vertex model commuting with the XYZ Hamiltonian;

(iv) a certain trivial vertex model ($w_5 = w_6 = w_7 = w_8$ and $w_1 = w_4$, $w_2 = w_3$ or $w_1 = w_3$, $w_2 = w_4$) of an essentially one-dimensional nature commuting with the isotropic Heisenberg Hamiltonian.

These cases exhaust all solutions of matrix dimension two, i.e. up to eight-vertex models. In this work I report a study of matrix dimension three which provides novel solutions of the Sutherland equation of the 15-vertex type and commuting with spin-1 (or three-component) Hamiltonians.

As is well known, there is an isomorphy between the local vertex operator A_n (also called the local L operator) of a vertex model and factorised relativistic S matrices of $(1+1)$-dimensional quantum field theories (Zamoladchikov 1979). Accordingly explicit

0305-4470/89/010025+07$02.50

realisations of such 15-vertex type solutions are provided by the factorised S matrices of Cherednik (1980) and those studied in a series of papers by Babelon, de Vega and Viallet (Babelon *et al* 1981, 1982, 1983a, 1983b, de Vega 1985, 1987).

In order to find operators A_n, B_n and $H_{n,n+1}$ satisfying equation (1), I proceed in the following manner. A_n is the local vertex operator, arising originally in the calculation of the partition function

$$Z = \mathrm{Tr}(T^M) = \mathrm{Tr}\left(\left(\mathrm{Tr} \prod_n A_n\right)^M\right). \tag{3}$$

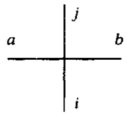
In the notation of de Vega (1985) and Takhtajan (1985) A_n is defined by

$$(A_n)^{ab} = \sum_{ij} A_{ij,ab} E_{ij}^{(n)} \tag{4}$$

where $E_{ij}^{(n)}$ is a matrix of the canonical basis of $\mathbb{R}^D \times \mathbb{R}^D$ and

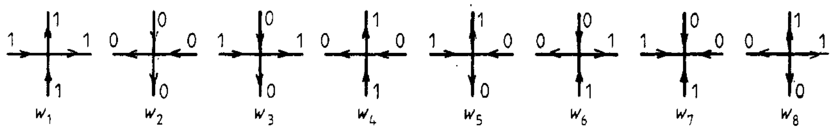
$$[A_n^{ab}]_{ij} = A_{ij,ab} \tag{5}$$

is the Boltzmann weight of a vertex

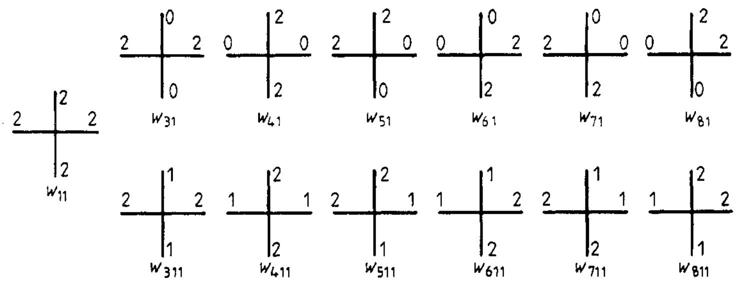


at site n of the lattice. D is the number of inner degrees of freedom and generalising the eight-vertex rule, only vertices with an even number of states of each type $(0, 1, 2, \ldots, D-1)$ are admitted.

For $D = 2$, the eight vertices are labelled as usual:



and for $D = 3$ there are, in addition, the following new vertices:



Accordingly, there are altogether 21 different vertices for $D = 3$:

$$\begin{array}{lll}
A_{11,11} = w_1 & A_{00,00} = w_2 & A_{22,22} = w_{11} \\
A_{00,11} = w_3 & A_{00,22} = w_{31} & A_{11,22} = w_{311} \\
A_{11,00} = w_4 & A_{22,00} = w_{41} & A_{22,11} = w_{411} \\
A_{01,10} = w_5 & A_{02,20} = w_{51} & A_{12,21} = w_{511} \\
A_{10,01} = w_6 & A_{20,02} = w_{61} & A_{21,12} = w_{611} \\
A_{10,10} = w_7 & A_{20,20} = w_{71} & A_{21,21} = w_{711} \\
A_{01,01} = w_8 & A_{02,02} = w_{81} & A_{12,12} = w_{811}.
\end{array}$$

For the sake of simplicity, I set $w_7 = w_8 = w_{71} = w_{81} = w_{711} = w_{811} = 0$, leaving 15 different non-zero vertex weights $w_1, \ldots, w_{611}$.

Starting from the observation that the stronger local condition

$$[A_n A_{n+1}, H_{n,n+1}] = 0 \tag{6}$$

has only trivial solutions, a set $\{B_n\}$ of auxiliary operators, to be determined in the course of the calculation, is introduced. Taking the trace of equation (1) imposes the following conditions on the B_n:

$$\begin{aligned} 0 &= \mathrm{Tr}(B_n A_{n+1} - A_n B_{n+1}) \\ &= \sum_{ij} (B_n^{ij} A_{n+1}^{ji} - A_n^{ij} B_{n+1}^{ji}). \end{aligned} \tag{7}$$

Hence the proper choice for B_n is $B_n^{ij} \sim A_n^{ij}$ implying that the quantity within brackets in (7) vanishes.

In the next step a suitable ansatz for the local interaction $H_{n,n+1}$ will be given. Having represented A_n and B_n in the canonical basis E_{ij}, it will be convenient to write $H_{n,n+1}$ in this basis, too:

$$H_{n,n+1} = \sum_{i,j,k,l} \lambda_{ij,kl} E_{ij}^{(n)} E_{kl}^{(n+1)}. \tag{8}$$

Obviously, any other basis will serve as well. For instance, one may use for $D = 2$ the three Pauli matrices and the unit matrix, and for $D = 3$ the eight SU(3) generators supplemented by the unit matrix, or equivalently three matrices belonging to the spin-1 representation of SU(2) together with six tensor matrices. I will calculate the anisotropy coefficients $\lambda_{ij,kl}$ in the canonical basis and present the result also in an SU(3) basis.

Substituting (4) and (8) into (1), one obtains the following system of equations for the unknown coefficients $A_{ij,ab}$, $B_{ij,ab}$ and $\lambda_{ij,kl}$:

$$\begin{aligned} \sum_{s_1,s_2,s_3=0}^{D} & [\lambda(s_1, d, s_2, f) A(c, s_1, a, s_3) A(e, s_2, s_3, b) \\ & - \lambda(c, s_1, e, s_2) A(s_1, d, a, s_3) A(s_2, f, s_3, b)] \\ &= \sum_{s=0}^{D} [B(c, d, a, s) A(e, f, s, b) - A(c, d, a, s) B(e, f, s, b)]. \end{aligned} \tag{9}$$

This is a Yang-Baxter type system of D^6 equations for (at most) $3D^4$ unknowns. I recall that the so-called Yang-Baxter or star-triangle equation is the local condition for different transfer matrices to commute

$$[T, T'] = 0 \tag{10}$$

and results in a similar system of equations for the corresponding coefficients, representing likewise D^6 equations for $3D^4$ unknowns (cf equation (9.6.8) of Baxter (1982)). A closer examination of (9) reveals that at least some of its solutions are also solutions of the Yang-Baxter equation and vice versa, so presumably (9) is related to the Yang-Baxter equation by some algebraic transformations. I will leave this conjecture open for further studies and proceed with the solution of the system of equations (9).

Recall that the complete solution for $D = 2$ (64 equations) has been given by Kasteleyn (1975). For $D = 3$ (729 equations) a geometrically motivated simplification will be introduced. The choice

$$\lambda_{ij,kl} = \delta_{ij}\delta_{kl}\lambda_{ii,kk} + \delta_{il}\delta_{jk}\lambda_{ij,ji} + \delta_{ik}\delta_{jl}\lambda_{ij,ij} - 2\delta_{ij}\delta_{kl}\delta_{ik}\lambda_{ii,ii} \tag{11}$$

leaves only 21 $\lambda_{ij,kl}$ different from zero and allows the formal identification of each $\lambda_{ij,kl}$ with an admissible vertex. If, furthermore, $\lambda_{ij,kl}=(\lambda_{ji,lk})^*$, the resulting Hamiltonian will be Hermitian.

Note that equations (9) are linear in the auxiliary quantities $B_{ij,ab}$ and in the anisotropies $\lambda_{ij,kl}$, but non-linear in the vertex weights $A_{ij,ab}$. Accordingly, first all $B_{ij,ab}$ and then all $\lambda_{ij,kl}$ can be eliminated, leaving a set of non-linear relations between the vertex weights $A_{ij,ab}$†.

It results that under the above provisos and after proper elimination of the $B_{ij,ab}$ and the $\lambda_{ij,kl}$, the 15-vertex solution requires the $A_{ij,ab}$ to satisfy the following six quartic relations:

$$w_{41}w_{31}w_{611}w_{511}=w_{411}w_{311}w_{61}w_{51} \tag{12a}$$

$$w_4w_3w_{611}w_{511}=w_{411}w_{311}w_6w_5 \tag{12b}$$

$$w_4w_3w_{61}w_{51}=w_{41}w_{31}w_6w_5 \tag{12c}$$

$$w_2^2w_{611}w_{511}-w_2w_6w_{611}w_{51}-w_2w_{61}w_5w_{511}-w_4w_3w_{61}w_{51}+w_6w_5w_{61}w_{51}=0 \tag{12d}$$

$$w_1^2w_{61}w_{51}-w_1w_5w_{61}w_{511}-w_1w_{51}w_6w_{611}-w_4w_3w_{611}w_{511}+w_6w_5w_{611}w_{511}=0 \tag{12e}$$

$$w_{11}^2w_6w_5-w_{11}w_5w_{61}w_{511}-w_{11}w_6w_{51}w_{611}-w_{41}w_{31}w_{611}w_{511}+w_{61}w_{51}w_{611}w_{511}=0. \tag{12f}$$

Every 15-vertex model fulfilling these relations commutes with a certain Hamiltonian, characterised by the following anisotropies:

$$\begin{aligned}
&\lambda_1\equiv\lambda_{11,11}=C+w_{61}w_{51}w_1x && \lambda_2\equiv\lambda_{00,00}=C+w_{611}w_{511}w_2x\\
&\lambda_3\equiv\lambda_{00,11}=C+w_6w_{51}w_{611}x && \lambda_4\equiv\lambda_{11,00}=C+w_5w_{61}w_{511}x\\
&\lambda_5\equiv\lambda_{01,10}=w_4w_{61}w_{51}x && \lambda_6\equiv\lambda_{10,01}=w_3w_{611}w_{511}x\\
&\lambda_{11}\equiv\lambda_{22,22}=2S-C+(w_{41}w_{31}w_{611}w_{511}-w_{61}w_{51}w_{611}w_{511})x/w_{11}\\
&\lambda_{31}\equiv\lambda_{00,22}=\text{arbitrary} && \lambda_{41}\equiv\lambda_{22,00}=2S-\lambda_{31}\\
&\lambda_{51}\equiv\lambda_{02,20}=w_{41}w_6w_5x && \lambda_{61}\equiv\lambda_{20,02}=w_{31}w_{611}w_{511}x\\
&\lambda_{311}\equiv\lambda_{11,12}=\lambda_{31} && \lambda_{411}\equiv\lambda_{22,11}=\lambda_{41}\\
&\lambda_{511}\equiv\lambda_{12,21}=w_{411}w_6w_5x && \lambda_{611}\equiv\lambda_{21,12}=w_{311}w_{61}w_{51}x\\
&\lambda_{711}=\lambda_{71}=\lambda_7=0 && (\text{or } \lambda_{21,21}=\lambda_{20,20}=\lambda_{10,10}=0)\\
&\lambda_{811}=\lambda_{81}=\lambda_8=0 && (\text{or } \lambda_{12,12}=\lambda_{02,02}=\lambda_{01,01}=0).
\end{aligned} \tag{13}$$

X, C, S and λ_{31} are arbitrary, representing, respectively, an additive and a multiplicative constant in the Hamiltonian and two free parameters.

The symmetries of Hamiltonian (8) with the coefficients (13) become more apparent in a basis formed by the eight SU(3) generators $T_1,\ldots,T_8$ and the unit matrix T_0. Here one has

$$H_{n,n+1}=\sum_{\gamma,\delta=0}^{8}\lambda_{\gamma\delta}T_\gamma^{(n)}T_\delta^{(n+1)} \tag{14}$$

† These cumbersome calculations were done with the algebraic program REDUCE on the SUN workstation of the Mathematics Department of the Universität des Saarlandes.

with the 21 non-vanishing coefficients $\lambda_{\gamma\delta}$ given by

$$\begin{aligned}
&\lambda_{00} = \text{arbitrary}\\
&\lambda_{11} = \lambda_{22} = x(w_{61}w_{51}w_4 + w_{611}w_{511}w_3)\\
&\lambda_{44} = \lambda_{55} = x(w_{611}w_{511}w_{31} + w_6w_5w_{41})\\
&\lambda_{66} = \lambda_{77} = x(w_{61}w_{51}w_{311} + w_6w_5w_{411})\\
&\lambda_{12} = -\lambda_{21} = \mathrm{i}x(w_{611}w_{511}w_3 - w_{61}w_{51}w_4)\\
&\lambda_{45} = -\lambda_{54} = \mathrm{i}x(w_{611}w_{511}w_{31} - w_6w_5w_{41})\\
&\lambda_{67} = -\lambda_{76} = \mathrm{i}x(w_{61}w_{51}w_{311} - w_6w_5w_{411})\\
&\lambda_{30} + \lambda_{03} = -(4/3)x(w_{61}w_{51}w_1 - w_{611}w_{511}w_2)\\
&\lambda_{38} + \lambda_{83} = -(2\sqrt{3}/3)x(w_{61}w_{51}w_1 - w_{611}w_{511}w_2)\\
&\lambda_{08} + \lambda_{80} = \text{arbitrary}\\
&\lambda_{33} = x(w_{61}w_{51}w_1 - w_{61}w_{511}w_5 - w_{51}w_{611}w_6 + w_{611}w_{511}w_2)\\
&\lambda_{88} = x(w_{61}w_{51}w_1w_{11} - 4w_{61}w_{51}w_{611}w_{511} + w_{61}w_{511}w_5w_{11}\\
&\qquad + w_{51}w_{611}w_6w_{11} + w_{611}w_{511}w_2w_{11} + 4w_{611}w_{511}w_{41}w_{31})/3w_{11}.
\end{aligned} \tag{15}$$

In geometrical language the 15 independent vertex weights span the unit cube in $\mathbb{R}^{15}$ and the Sutherland equation is fulfilled on the hypersurface or hypersheaf defined by (12). It will be convenient to look at this hypersheaf along some diagonal cut. This is equivalent to imposing special symmetries to the vertex weights. I will give three examples, all of which may be considered as genuine extensions of the six-vertex model to the three-state case.

(i) The solution with $\mathbb{Z}_3$ symmetry ($A_{ij,kl} = A_{i+n,j+n,k+n,l+n}$ for $n \in \mathbb{Z}_3$) is

$$\begin{aligned}
&w_1 = w_2 = w_{11} = A \qquad w_3 = w_{41} = w_{311} = A_3 \qquad w_4 = w_{31} = w_{411} = A_4\\
&w_5 = w_{61} = w_{511} = C_5 \qquad w_6 = w_{51} = w_{611} = C_6\\
&A^2C_5C_6 - AC_6^3 - AC_5^3 - A_3A_4C_5C_6 + C_5^2C_6^2 = 0.
\end{aligned} \tag{16}$$

(ii) The solution with link-state symmetry is

$$\begin{aligned}
&w_1 = w_2 = w_{11} = A \qquad w_3 = w_{31} = w_{311} = A_3 \qquad w_4 = w_{41} = w_{411} = A_4\\
&w_5 = w_{51} = w_{511} = C_5 \qquad w_6 = w_{61} = w_{611} = C_6\\
&(A - C_5)(A - C_6) - A_4A_3 = 0.
\end{aligned} \tag{17}$$

(ii) The solution of de Vega (1987) is

$$\begin{aligned}
&w_1 = w_2 = w_{11} = A \qquad w_3 = w_{41} = w_{311} = A_3 \qquad w_4 = w_{31} = w_{411} = A_4\\
&w_5 = w_{51} = w_{511} = C_5 \qquad w_6 = w_{61} = w_{611} = C_6\\
&(A - C_5)(A - C_6) - A_4A_3 = 0.
\end{aligned} \tag{18}$$

Here the constraint equations result from (12) and define, respectively, a hypersurface in the space of vertex weights. This should be included in the definition of the vertex weights by means of some suitable parametrisation. It turns out that for (16) the following parametrisation in terms of hyperbolic functions is appropriate:

$$\begin{aligned}
&A = 1 \qquad A_3 = g(\gamma)\sinh\theta/\sinh(\theta+\gamma) \qquad A_4 = g^{-1}(\gamma)\sinh\theta/\sinh(\theta+\gamma)\\
&C_5 = \exp(\theta/3)\sinh\gamma/\sinh(\theta+\gamma) \qquad C_6 = \exp(-\theta/3)\sinh\gamma/\sinh(\theta+\gamma).
\end{aligned}$$

In (17) and (18), we are left with a simple quadratic form and can thus transform to main axis and construct a parametrisation with four independent parameters θ, γ, δ and t:

$$A = s + \sinh(\theta+\gamma) \qquad A_3 = \sinh\theta\,(1 - \sinh\gamma\sinh t)$$

$$A_4 = \sinh\theta\,(1+\sinh\gamma\sinh t)$$

$$C_5 = s + \sinh\gamma\,(\cosh\theta + \sinh\theta\cosh t) \qquad C_6 = s + \sinh\gamma\,(\cosh\theta - \sinh\theta\cosh t)$$

or, using a restriction to a two-parameter submanifold with $s = t = 0$:

$$A = 1 \qquad A_3 = \sinh\theta/\sinh(\theta+\gamma) \qquad A_4 = \sinh\theta/\sinh(\theta+\gamma)$$

$$C_5 = \mathrm{e}^{\theta}\sinh\gamma/\sinh(\theta+\gamma) \qquad C_6 = \mathrm{e}^{-\theta}\sinh\gamma/\sinh(\theta+\gamma).$$

Both are also standard solutions of the parametrised Yang–Baxter equation (Cherednik 1980, Kulish and Sklyanin 1982, de Vega 1987, de Vega and Karowski 1987), thus exhibiting once more the intimate connection between the Sutherland equation and the Yang–Baxter equation. The exact integrability follows at once, for, if θ is a parameter along the hypersurface, and

$$[T(\theta), \mathscr{H}(\theta')] = 0$$

then there is an infinite number of integrals of motion

$$\left.\frac{\partial^k}{\partial\theta^k}\log T(\theta)\right|_{\theta=\theta'}$$

for the Hamiltonian.

In conclusion I have presented a new 15-vertex solution to the Sutherland equation†. Together with the known six- and eight-vertex solutions, this probably exhausts all solutions with up to 15 vertices. From the body of knowledge collected during recent years on the solutions of the parametrised Yang–Baxter equation (Belavin 1981, Bazhanov 1987, Akutsu and Wadati 1988), which are all parametrised by either (i) elliptic, (ii) trigonometric or hyperbolic, or (iii) rational functions, one might expect that this 15-vertex model corresponds to a 21-vertex model parametrised by elliptic functions at criticality.

I would like to thank H Altpeter-Rüb for an introduction to REDUCE and to the use of the SUN workstation, H-G Rück and A Holz for helpful discussions and the latter also for a critical reading of the manuscript.

References

Akutsu Y and Wadati M 1988 *Commun. Math. Phys.* **117** 243–59
Babelon O, de Vega H J and Viallet C M 1981 *Nucl. Phys.* B **190** 542–52
—— 1982 *Nucl. Phys.* B **220** 266–80
—— 1983a *Nucl. Phys.* B **220** 13–34
—— 1983b *Nucl. Phys.* B **220** 283–301
Bamberg J and Schmitt H 1988 *Solid State Ionics* **26** 303–6

† The 15-vertex model can be used in a simulation of the order–disorder transition observed in hydrogen molybdenum bronze $H_{1.6}MoO_3$ (Bamberg and Schmitt 1988).

Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)
Bazhanov V V 1987 *Commun. Math. Phys.* **113** 471-503
Belavin A A 1981 *Nucl. Phys.* B **180** 189-200
Cherednik I V 1980 *Teor. Mat. Fiz.* **43** 117-9 (1980 *Theor. Math. Phys.* **43** 356)
de Vega H J 1985 *Preprint* LPTHE 85/54
—— 1987 *J. Phys. A: Math. Gen.* **20** 6023-6
de Vega H J and Karowski M 1987 *Nucl. Phys.* B **280** 225-54
Kasteleyn P W 1975 *Fundamental Problems in Statistical Mechanics* vol 3, ed E G D Cohen (Amsterdam: North-Holland)
Kulish P P and Sklyanin E K 1982 *J. Sov. Math.* **19** 1596-620
Sutherland B 1970 *J. Math. Phys.* **11** 3183-6
Takhtajan L A 1985 *Exactly Solvable Problems in Condensed Matter and Relativistic Field Theory* (*Lecture Notes in Physics* **242**) ed B S Shastry, S S Jha and V Singh (Berlin: Springer)
Zamolodchikov A B 1979 *Commun. Math. Phys.* **69** 165-78